GCSE 2D Geometry and Angles.

Welcome to the 2D Geometry and Angles text. This independent text will cover everything needed for your course and more, so be prepared to learn, meaning you will require no prior knowledge of other parts of the syllabus. It will give you the foundations needed for any A-level Maths course, or at the very least the knowledge to complete any geometry question on your paper accurately.

We will cover topics such as finding angles in triangles contained within circles, the trigonometric functions, angle theorems, and their proofs. We will introduce the most common 2D shapes, and then start with some introductory angle theorems, so that we may prove certain properties of the shapes we introduce. We will then move on to areas of shapes, including the cosine rule, the area of a triangle, and more.

Contents

Chapter 1 – Key shapes in 2D

A good point to begin would be by introducing the shapes we will be working with. There are several shapes you will need to know, and although this of course seems very trivial (after all, you know what a square, a triangle and a circle are..). However, in an exam not a single person will tell you what a quadrilateral, a decahedron, or what the difference between a sphere, cone and pyramid is. For this reason, we aim to first make you comfortable with all the basic shapes you might meet in the exam.

Definition 1 (*circle)* – A circle is the set of all points which are a distance r from its centre.

Most people know what a circle is (but if not, and you have any trouble understanding the definition of a circle here, please don't hesitate to get in contact via the contact page), but the above description is more in depth. It highlights a key feature of the shape; that every point on a circle is a distance r from the centre of the circle.

We call the distance r the **radius** of the circle. The length running around the edge of the circle is called the circle's **circumference**. If we multiply r by two (which we write as $2r$), we obtain the **diameter** of the circle.

Example

If we let $r = 3$, then the resulting circle is the set of all points which are *all* 3 units away from the centre (simultaneously). Ie,

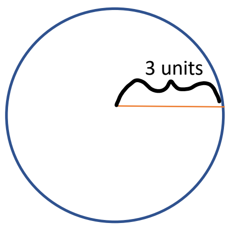


Figure 1

The blue part of the above picture would be the circumference, the radius is 3, and the diameter of the circle would be $2 \times 3 = 6$.

Definition 2 (*triangle*) – A <u>triangle</u> is the shape formed by three straight lines joined together.

For example, the following is a triangle.

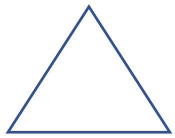


Figure 2

There are four types of triangle that you will meet at GCSE; an **equilateral**, **isosceles**, **scalene**, and **right angled** triangle.

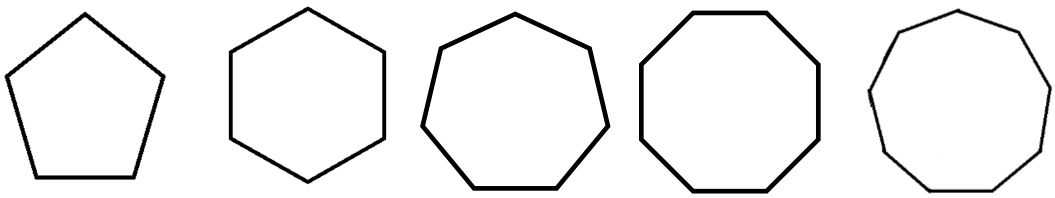
- <u>Equilateral</u> triangles are a triangle where each side is the same length.
- <u>Isosceles</u> triangles are triangles where two sides are the same length, and the third line is **not** the same length as the other two.
- <u>Scalene</u> triangles have three different sides, ie no side has the same length as any other.
- A <u>right angled</u> triangle is any triangle that has a $90°$ angle between any two of its sides.

Definition 3 (*rectangle*) – A <u>rectangle</u> is any shape with four straight sides, and the angle between any two intersecting sides is $90°$.

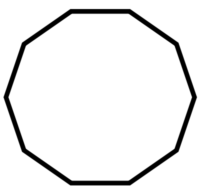
A special type of rectangle that you will be aware of is a **square**. It has four sides, and the angle between its sides is $90°$ at each corner.

Figure 3 and 4

The remaining 2D shapes that you need to know are the following,



	Pentagon	**Hexagon**	**Heptagon**	**Octagon**	**Nonagon**
Number of sides:	**5**	**6**	**7**	**8**	**9**



Decagon

Number of sides:	**10**

We call these six shapes **regular polygons**. The reason for this is that in each shape the sides are the same size, and therefore the angles between any two sides is the same for each pair of intersecting sides.

Definition 3 (*rhombus*) – A <u>rhombus</u> is the shape produced by sliding the top side of a rectangle horizontally (either left or right).

Note that this definition starts from a square and always gives you a rhombus, but this is not the ***only*** *way to make a rhombus from a square!*

Example,

Figure 5

Figure 6

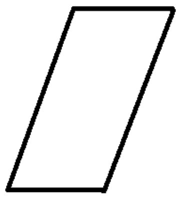


Figure 7

Here, all the Rhombus have been created by shifting the top side of each rectangle to the *right*. If we had moved the line to the *left*, we would still have ended up with a rhombus.

In this short introduction to the 2D shapes we will deal with, we spoke of *sides*. However, there is some more specific terminology that you will need to be comfortable with.

Definition 4 (*vertex/vertices*) – The point where any number of lines meet are called <u>vertices</u>.

For example, a triangle,

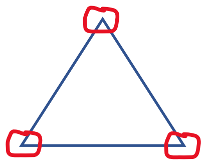


Figure 8

has three vertices (marked in red above). Another, less formal, word for vertex would be a *corner*.

Definition 4 (*Perimeter*) – Consider a shape such as that in figure 5. The <u>perimeter</u> of the shape is the sum of the lengths of all the sides of the shape.

Example,

Consider the following shape,

Figure 5

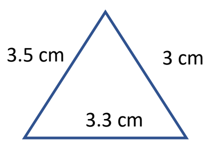


Figure 9

The perimeter would be $3.5 + 3 + 3.3 = 9.8cm$. We simply add up the lengths of all the sides to obtain the perimeter. In the next chapter we will deal with angles. We will introduce and explain some of the important theorems you need to know for the exam, and use these theorems to derive further theorems. Do not be alarmed as all will be explained!

Chapter 2 – Length and angles

This section will introduce a lot of theory about angles. Do feel free to get yourself a tea, as there will be a lot of reading!

Definition 5 (*Angle*) – Consider two intersecting lines, such as below (figure 7),

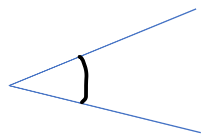


Figure 10

If we now overlay a circle of radius r, with its centre at the vertex (ie, the point at which the two lines intersect), we obtain the following diagram,

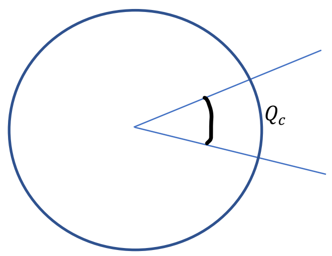
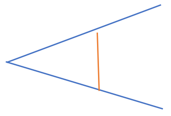


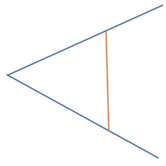
Figure 11

The <u>angle</u> between the two lines is equal to the circumference contained between the two lines (labelled as Q_c on figure 11), divided by r, the radius of the circle.

In layman's terms, if we know the "*angle*" between the two lines, then for each point a on the line A, we know the minimum distance between a and the line B. This is definition may seem obscure, but it actually describes something we all know to be very obvious. As a specific case of this, take a pair of scissors. If we "open" the scissors, we make the two ends of the scissors further apart, and they look similar to this,



Wide



Even wider

Note how the same two points on the two scissor ends become further apart as we open the scissors more? In general, if we have two intersecting lines, the angle between them is defined by the minimum distance between some point on one line and the second line.

The above definition, as you will learn in higher level courses, measures *angles* in a unit called a **radian**. We will only be looking at angles in a unit called a *degree* in this text. In order to convert from radians, as above, to degrees, we use the fact that π radians is equal to 180 degrees. Therefore, one radian is equal to $\frac{180}{\pi}$ degrees. So, m radians is equal to $\frac{180}{n} \times m$ degrees.

Theorem 1 – There are 360 degrees in a full circle. Or, equivalently, there are 2π radians in a full circle.

Proof

If we have two overlapping lines, such as the following,

Note that it looks like only one line as they overlap. Here, we can actually fit a full circle between the lines (or no part of a circle, depending on the part you consider), so the "amount" of circumference is equal to the full circumference of the circle, $2\pi r$.

It follows, then, that, from definition 5, the angle between the lines is $\frac{2\pi r}{r} = 2\pi$. Therefore, in degrees, we have that the angle is equal to $2\pi \times \frac{180}{\pi} = 2 \times 180 = 360°$.

We will use this fact many times throughout this text, so make sure you know that a full circle has $360°$.

<u>The three trigonometric functions</u>

Just as we can have functions of numbers, ie $y = 2x + 3$, we can also have functions that act on the above measurement, angles. Three such, very important, examples, are the *trigonometric functions*.

You should be familiar with the concept of a graph, where a coordinate (x, y) is located x units along and y units high. If we consider a circle of radius 1 in a graph, then for any angle, by definition we know what the coordinates of the point in that direction is.

Being more precise, if we calculate the angle between the x-axis and the radius (going anticlockwise), So, you see, if we know how "open" they are, we also "know" how wide the scissors will be. Consider the following diagram:

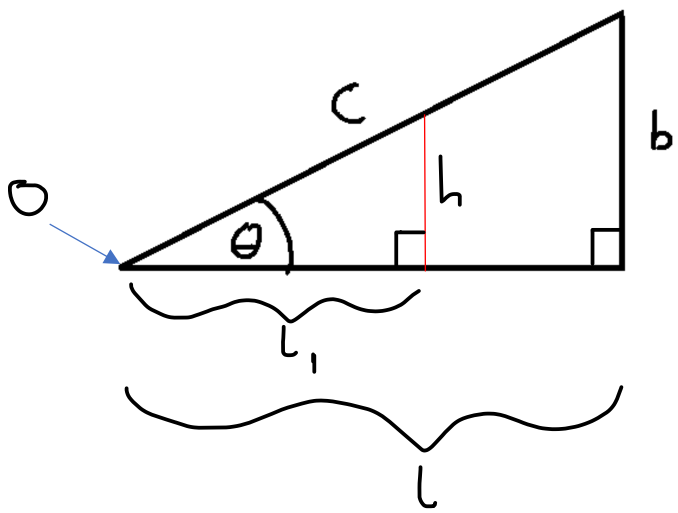


Figure 12

In other words, we know that for two lines that are θ degrees apart, we can work out the vertical distance h between the two lines at some distance x from the point of intersection (recall that the point of intersection is the point at which the two lines meet, marked O in figure 12). Well, why not define this as a function? If you agree, you wouldn't be the first to ask that question; we call this the **tan** function. Look back at figure 12. If we consider the smaller triangle, which we label as figure $12b$,

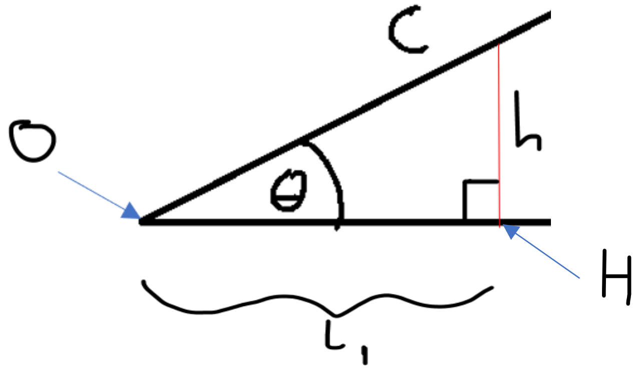


Figure 12b

given the angle θ and the distance from O to the point H, l_1, it follows from our discussion above regarding the scissor metaphor, that we can work out the value of h. To do so, we use the tan function, written $\tan(\theta)$, and is defined as follows,

$$\tan(\theta) = \frac{h}{l_1}$$

Where h and l_1 are taken from figures 12 and $12b$. Now, multiplying both sides of the equation by l_1, we obtain

$$l_1 \times \tan(\theta) = h$$

Now, you may have noticed that we do not actually know the value of $\tan(\theta)$, however, when we plug $\tan(\theta)$ into a calculator, it works out its value for us using either something called "Taylor series", or an algorithm. Having worked out the value of $\tan(\theta)$, the calculator can then use the above formula to find the value of h.

We now consider for the final time figure 12. If we knew the value of θ and the value of C, then we can calculate either the value of l or b using the $cosine$ and $sine$ functions. The $cosine$ and $sine$ functions are defined in a similar way to the $tangent$ function:

- $sine(\theta) = \frac{b}{C}$
- $cosine(\theta) = \frac{l}{C}$

And again, our calculator works out the values of $sine$ and $cosine$ for us. Now, it is important to state a few things about figure 12. Note that the triangle in figure 12 *and* in figure 12b are all **right-angled** triangles. This is important, as the above equations for $sine$, $cosine$ and $tangent$ are **only** valid when we have a right-angled triangle (a right angle is where there is an angle of $90°$ between any two sides of the triangle). This means we can only use the above formulas to find the missing length(s) if one of the angles inside the triangle is $90°$.

The final part of this chapter refers to a different type of problem. Recall the formula above,

$$\tan(\theta) = \frac{h}{l_1}$$

Now, say we know the values of h and l_1, and we want to use this to work out the angle which is opposite to the side h. Recall from your studies of **inverse functions** that, for some function, we can "reverse" them. In other words, if $\tan(x) = y$ for some value of x, then we can "reverse" the function and go from y to x. The same is true for the trigonometric functions, we obtain,

- $\tan^{-1}\left(\frac{h}{l_1}\right) = \theta$
- $\cos^{-1}\left(\frac{l_1}{c}\right) = \theta$
- $\sin^{-1}\left(\frac{h}{c}\right) = \theta$

And again, our calculator works out the values of $\sin^{-1}(x)$ using advanced algorithms, so we just need to put these values into our calculator.

Chapter 3 – Angle theorems

Firstly, having defined what an angle is in the previous chapter, we introduce the *types* of angle you will meet throughout your course. The type of angle simply depends on the size of the angle between two lines. In increasing order, we have the following types of angle,

1) **Acute Angle**: $\theta < 90°$.
2) **Right Angle**: $\theta = 90°$.
3) **Obtuse Angle**: $90° < \theta < 180°$.
4) **Straight Angle**: $\theta = 180°$.
5) **Reflex Angle**: $180 < \theta < 360°$.

Now, you may ask, what happens if the angle is equal to 360 or zero? Well, if the angle between two lines is zero, then this is less than $90°$, and so it is an acute angle. Recall that there are $360°$ in a full circle, so if we go all the way round, then we end up back where we started, and it follows that an angle of $360°$ between two lines is the same as an angle of $0°$ between them. Therefore, an angle of $360°$ is equivalent to an angle of $0°$, and thus is also an acute angle.

So, now we know the name for each size of angle, we can be more precise when introducing angle theorems, by stating clearly for which types of angles the theorem is valid.

The first set of theorems we will look at (and prove!), are aimed at finding the angle between two lines if we know the angle between two similar lines. The reason these theorems are important is that, rather than measuring each angle twice, if we can measure one angle and mathematically derive the second angle using the first, then we will be much more accurate than measuring two angles separately. This has applications in Architecture and Engineering, where accuracy is often important.

However, before we delve into the angle theorems, we need to learn a bit more about triangles. More specifically, about right-angled triangles, as we will use this to prove the angle theorems.

Definition 5 (*Congruent*) – Two shapes, shape A and shape B are said to be congruent if, from shape A, we can obtain shape B via some combination of the following:

1. A reflection.
2. A rotation.
3. A translation.

So, the following triangles are congruent,

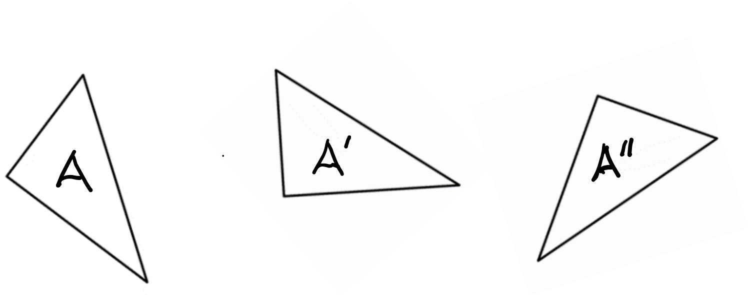


Figure 13 - (Assume all sides have the same length and all angles are the same in A, A' and A''.)

One consequence of this is that, if two triangles are congruent, then we know that their corresponding sides are the same length, and hence their angles must also be the same. We will use this fact when we go through the angle theorems.

Theorem 1 – Consider two intersecting lines, A and B, which meet at a point c, as shown below,

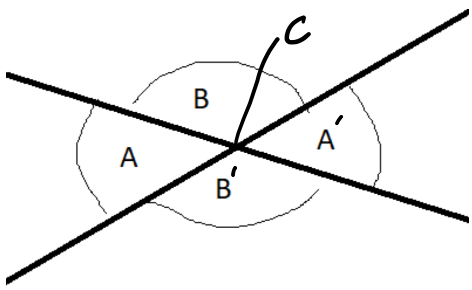


Figure 14

Then, as in the picture, opposite angles are equal. That is, $A' = A$ and $B' = B$.

Proof

Imagine that the point c has coordinates $(0,0)$, and the two lines are given by the equations $y = -ax$ and $y = bx$, where a and b are positive numbers.

We wish to show that the distance between the two lines is the same at $x = x_1$ and $x = -x_1$. At $x = x_1$, the distance is given by $h = \sqrt{\left(bx_1 - (-ax_1)\right)^2} = \sqrt{(bx_1 + ax_1)^2} = bx_1 + ax_1 = (b + a)x_1$.

For $x = -x_1$, we have $h = \sqrt{\left((-bx_1) - (ax_1)\right)^2} = \sqrt{(-bx_1 - ax_1)^2} = \sqrt{\left((-1)(bx_1 + ax_1)\right)^2}$

$$= (a + b)x_1$$

So, we have shown that $h_1 = h_2$. We also know that the two sides have the same length, because they are the same equation in different directions ($\pm x_1$). This means that, because the sides are the same length, and in the same order, then the corresponding angles will be the same. This proves the theorem.

Theorem 2 – Consider two angles as drawn below,

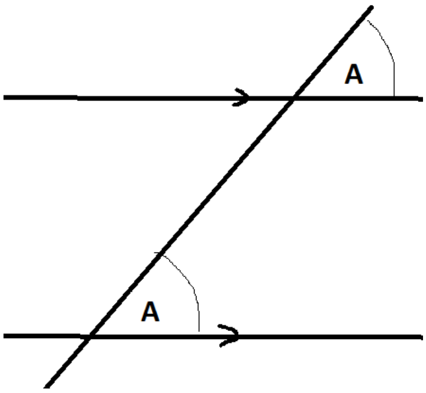


Figure 15

Where the two horizontal lines are parallel to each other, and the third, diagonal, line intersects both horizontal lines. Then, the two angles, both labelled A, are equal. We will not prove this here, as it is just the same lines meeting twice, and so the angle must be the same in both situations, as the angle does not depend on the point of intersection.

Making use of this fact, we can show that

As a consequence of the two theorems above, as well as the fact that a full circle is $360°$ (and hence half a circle is $\frac{360°}{2} = 180°$), we know that both angles adjacent to angle A must be equal to $180 - A$ degrees. Then, using theorem 2, we have that **Must now add this consequence to the picture above.**

Theorem 3 – Consider figure 15. If we add another label for angle B, which is adjacent to angle A;

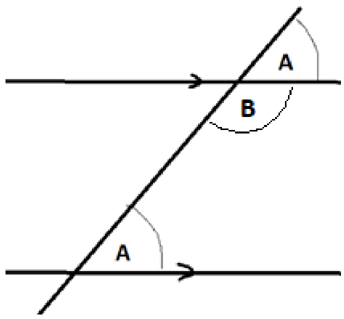
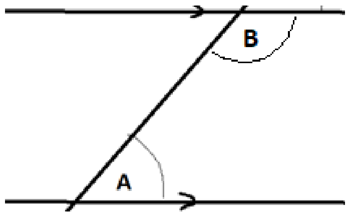


Figure 15a

Then, $A + B = 180°$.

This should be clear from the fact that $A + B$ lies along a straight line by theorem two, and the fact that we can only fit half the circumference of a full circle around the same side of a straight line (so a straight line has $180°$). This means that whenever we have two parallel lines, and a diagonal line joining them, then the two angles on the same side of the diagonal line and in-between the two parallel lines, will equal $180°$;



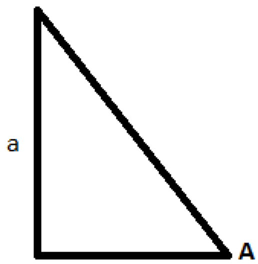
$$A + B = 180°$$

This concludes the more basic angle theorems involving parallel lines. We now have what we need to prove a fact you may already know. That the sum of the angles at each vertex of a triangle equals $180°$.

Theorem 4 – The sum of the angles at each vertex of a triangle is equal to $180°$.

Proof

Imagine an arbitrary triangle. It, by definition, has 3 vertices, where each vertex is opposite to one side. Now, if we choose a vertex A, and consider the corresponding side, a,



Then we can draw a line parallel to the line a which intersects the vertex $\boldsymbol{A}$, and we obtain the following,

Now, the

Allied angles

Corresponding angles

Alternate angles

Chapter 2 – Length and areas in 2D

We gave an example above (*Figure 1*) of a circle with radius 3 units. However, what does this mean exactly? In this chapter we will shortly introduce **length**, which will then lead us on to **area**.

Imagine you face forward, and then walk forward 3 meters. Or, similarly, imagine the path taken by a car going around a corner. Before we can model these phenomena mathematically, we need to have some quantities to describe how far you, or the car, have travelled. In order to describe this rather basic property of a path taken, we have developed different *units* of measurement. The most common set of units used in Mathematics is the empirical measurements. You have no doubt heard of them, some examples are the **kilometre**, **meter** and **centimetre**.

Now, how does this allow us to find the areas of the shapes we saw above? Well, we will start by explaining how to find the area of the shapes we will come across during the GCSE maths course.